The Invisible Hand in Motion

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1 Introduction

One area which I have struggled with most is the stability properties of competitive equilibrium or what one may describe as the effectiveness of the Invisible Hand in Motion. From the time of Adam Smith, there was a belief that the so called Invisible Hand, the market force, should be successful in attaining equilibrium or a matching of demand and supply. That this force was believed to be as mysterious and powerful as the Force which Luke Skywalker is supposed to have battled the odds with, is perhaps not very surprising. What is surprising however, is the almost complete neglect of this area by economic theorists.

To briefly recapitulate, the problem was discussed by Walras in the 19th century (1874) but then he assumed the problem away; it was redefined by Hicks in Value and Capital in 1940 and later by Samuelson in the Foundations in 1950's and subjected to searching and probing analyses by Arrow and Hurwicz and McKenzie in the early 1960's[1]; a masterly survey was produced by Takashi Negishi in 1962; but around the same time two papers, one by Scarf (1960) and the other by Gale (1963), showed that things were not as satisfactory as they might have been taken to be.

Briefly the situation was as follows: if demand and supply do not match, it was entirely expected that prices would adjust in the direction of excess demand (which is demand - supply) and would thus remove any vestige of mismatch. Even to this day, a sudden spurt of demand is associated with a rise in prices. The intuitive appeal of this story is however far more strong than its actual effectiveness.

Since I shall come back to this aspect later, let me quickly move on to describe why the problem received attention from only one class of economists. I should hasten to add that I am not really skilled in such historical matters and I can add a rather cursory approach to the entire question; much of classical economics used

[1] The paper by McKenzie (1960), contained the most general treatment of the subject.

the assumption of a single unproduced factor of production: any undergraduate in economics would or should be able to say that under such a situation, prices were determined solely from production conditions and there was no role for demand to play in the determination in price. If demand exceeded supply then momentarily there would be a mismatch, price would rise, profits would soar, attract more firms into the industry and supply would rise and then ultimately the price would fall back to its original value. Thus the price would be determined and be affected only if technology altered. The above explanation is however a purely partial equilibrium one and there are important deficiencies from a general equilibrium point of view. For instance, as soon as the price rises, the immediate effect is to shut off the production of the other goods (recall the unique price configuration which allowed the production of all goods) and several other firms go out of business. However it is possible to construct a pricing rule rather like a mark-up theory of pricing in such contexts[2].

As soon as one introduces the idea of other factors of production which cannot be produced, the story needs to be altered and demand comes to play some role. In particular, the role would be so central as to turn the entire story around and no longer could one assume the nice convergence to equilibrium; this aspect was clearly shown by the examples of Gale and Scarf i.e., there would be no reason to expect that the so-called invisible hand would work. In fact, a commentator no less eminent than Edmond Malinvaud in 1993, while celebrating 40 years of the proof of existence of competitive equilibrium would indicate that this was one of failures of economic theory.

In spite of this state of affairs, the reaction of the profession has been unsatisfactory. It may appear that most theorists were of the opinion that one ought to just forget the matter. Apart from this attitude there was an additional problem: any research in this area was not encouraged; this lack of encouragement

[2]See Mukherji (1974)

stemmed from the reaction of most editors of journals refusing to even consider articles in the area. That some persisted in their efforts in this direction must be attributed to their sheer cussedness. This general attitude[3] would have been one of the 'strange but true' items with marginal academic or pedagogic interest were it not for the fact that during the late eighties something happened which changed the situation drastically. The competitive equilibrium became the centre piece of the only economic paradigm relevant for formulating policy.

This tendency of considering economic policies based on the workings of the Invisible Hand had been questioned by Hahn (1984), (p. 308): "Now one of the mysteries which future historians of thought will surely wish to unravel is how it came about that the Arrow-Debreu Model came to be taken descriptively; that is as sufficient in itself for the study and control of actual economies" . But now with the clear winner amongst economic doctrines being the competitive markets and their working, it was even more inexplicable that very few undertook a close scrutiny of the conditions under which competitive markets work; in fact economists such as Sen (1999) (p. 112) advocated, "The need for critical scrutiny of the standard preconceptions and political-economic attitudes have never been stronger". And the strange and unresolved question of the stability of competitive equilibrium, the effectiveness of the Invisible Hand in Motion was never re-examined! This then is the background against which I would like to present some of my findings in the last few years.

There were several reasons why the study of stability problems was in disrepute. A strong contender for the primary position amongst them was the question of the manner in which the equation of motion of the invisible hand was written. The equation of motion was represented as a system of differential equations of the

[3]There have been some notable exceptions of course; consider for example, the contributions of d'Aspremont and Dreze (1979) and Champsaur, Dreze and Henry (1977).

type:

$$\dot{p} = \Phi(Z(p))$$

where $p, Z(p)$ represent prices and excess demands respectively and $\Phi(.)$ is some sign-preserving function. It was difficult to see whose behavior was being described by these equations and why that person or decision maker was behaving in such a fashion. The answer that this was how the Invisible Hand worked, although was found acceptable during the time of Smith did not find too many takers in recent times.

Fortunately, the recent experiments of Charles Plott of Caltech and his associates tackled this question directly[4] :"Experimentalists have discovered that classical models, which are based on an assumption of tatonnement, have remarkable explanatory power even when applied to the non-tatonnement, continuous, double auction markets" . The classical tatonnement is precisely what we have defined just now. Consequently, given the results of the experiments conducted by Plott and his associates, we may address ourselves to the analysis of the classical tatonnement, even though we are not sure whose behavior is being analyzed; thanks to the experimental results, we know that this analysis will be a good predictor of what to expect.

As we have mentioned above, the primary point of concern about the classical tatonnement lay in the paucity of results: these were difficult to come by and involved restrictions which appeared to be unwarranted. Technically speaking, what was at stake was the convergence of the solution to the system defined earlier[5].

Before we start looking at the convergence, it is best to mention that the

[4]See Anderson et. al.(2004) and Hirota et. al. (2005), Divergence, Closed Cycles and Convergence in Scarf Environments: Experiments in the Dynamics of General Equilibrium Systems, Caltech Working Paper.

[5]I should say that the motion could be thought of as being in discrete time as well, but we shall confine ourselves to the continuous time version of the Invisible Hand: it makes good sense to keep the steps of the Invisible Hand invisible as well.

classical tatonnement was introduced by Walras and essentially involved two things: first that price movement was in the direction of excess demand and second that transactions occur only at equilibrium so that along the path of price adjustment, no transactions took place. We have shown elsewhere[6] that it is possible to relax both of these conditions and still consider the convergence to equilibrium. But to-day I shall confine myself to the classical price adjustment process.

It may be important to point out that during the mid-seventies, a series of papers by Sonnenschein and Debreu established the fact that the two properties of excess demand viz., homogeneity of degree zero in prices and Walras Law were not restrictive enough. In fact, if these papers had appeared say five years earlier, I would have been in deep trouble since then I would not have ventured into the dissertation that I wrote in the early 70's. For these papers established that almost any set of functions which satisfied these restrictions could be thought of as being excess demand functions. This was the first type of result which has been termed to belong to the class of 'Anything Goes' results in the literature. But examples of instability were there from the very beginning. Thus it was not possible to infer anything meaningful from the assumptions of rational behavior by decision makers: the source of meaningful theorems that Samuelson had advocated in the Foundations, proved to be woefully inadequate in this respect. In fact, this lack of generality of results was not peculiar to the stability problem; many developments in different branches of economic theory were dependent on special forms and restrictions as well. The only difference that I can find is that in these other theoretical developments, no one ever owned up to the fact that their conclusions depended on there being a straight line demand curve or on a special form of the utility function; on the contrary, in the area of general equilibrium, which was primarily responsible for analysing the workings of the 'Invisible Hand', researchers were trying to show how special the theory was and how careful one must be to draw

[6]Mukherji(1995), (1990).

the 'correct' inferences. My faith in these latter norms has remained strong. This belief led me to investigate whether any set of meaningful economic conditions could be identified as being 'necessary' for the stability of equilibrium; such conditions were found and the fact that these are substantial requirements were noted[7]; it is only relatively recently[8] that the result was rescued from its mimeographed and privately circulated status.

There is another reason for general results being scarce; this has to do with the state of mathematics for the system of differential equations that we wrote down; a problem has been that economists generally conceive of their domain of discussion as one involving n variables; and to be respectable, you must not specify what n should be except to say that $n \geq 2$. Specifying n to be exactly 2 or 3 would immediately blacklist you; I went along with this peer-group pressure till I encountered Richard Day at a Conference of the Econometric Society. He advised me to study n = 1 first, carefully and then once that has been mastered, go to n = 2; and so on; beyond 3, it becomes very difficult to pin things down. Encouraged by this advice, I started to analyze smaller systems and that paid some dividends.

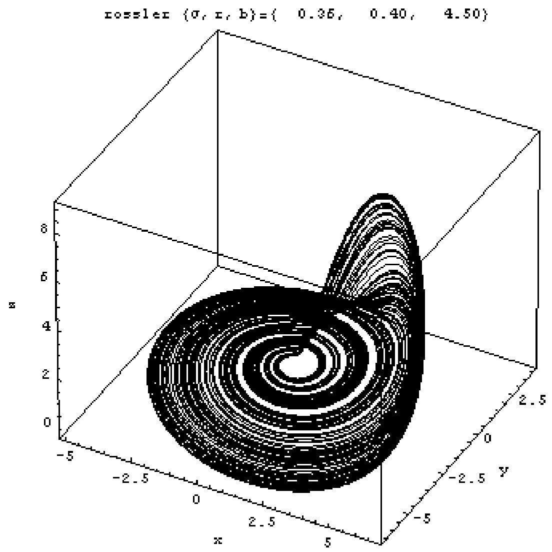
There is another deeper reason for Day's advice and these are related to the results of Stephen Smale, recipient of the Fields Medal[9] who showed that if the system of differential equations involved three equations, or more, then rather

[7] Mukherji (1989).

[8] See, McKenzie (2002), p. 102.

[9] Smale was awarded a Fields Medal at the International Congress at Moscow in 1966. One of Smale's impressive results was his work on the generalised Poincaré conjecture. The Poincaré conjecture, one of the famous problems of 20th-century mathematics, asserts that a simply connected closed 3-dimensional manifold is a 3-dimensional sphere. The higher dimensional Poincaré conjecture claims that any closed n-dimensional manifold which is homotopy equivalent to the n-sphere must be the n-sphere. When n = 3 this is equivalent to the Poincaré conjecture. Smale proved the higher dimensional Poincaré conjecture in 1961 for n at least 5. (Michael Freedman proved the conjecture for n = 4 in 1982 but the original conjecture remains open, as far as I am aware.)

Figure 1: The Rössler Attractor



arbitrary behavior is possible in the sense that the attractors, the sets to which the solutions converge, could be arbitrary. This fact is often neglected by economists who treat dynamic results in lower dimensions with disdain.

There is yet another problem that needs to be overcome viz. that of non-linearity of the functions concerned; excess demand functions must necessarily be non-linear: this follows from the Walras Law. Even the straightforward forms of non-linearity in a differential equation system may lead to very complicated dynamics Consider, for example the equations defining the so called *Rössler* Attractor:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ z \end{pmatrix} = \begin{pmatrix} -y - z \\ x + \sigma y \\ rx - bz - xz \end{pmatrix}$$

where σ, r, b are positive constants.

The solution to the above system has a very strange long-term behavior. See for example the Figure 1 where the values $\sigma = 0.36, r = 0.4, b = 4.5$ have been used. Notice that the extent of non-linearity is quite mild: the non-linear term appearing only in one equation. This is a well known system and the attractor is called the Rössler Attractor. Thus on the basis of dynamical systems alone, we have **another** source of "Anything Goes". On several counts, therefore, general results are not possible. I do not think that these aspects have been as well appreciated as they ought to be.

2 Preliminary Considerations: The Single Market

To motivate our analysis, we consider first of all, the case of two goods so that there is a single market. In a single market, the notion of a locally stable equilibrium is identified with the slope of the excess demand curve being negative at the equilibrium. But there does not seem to be any easy identification of global stability with any property even in such a simple set-up.

Consider the following example due to Gale (1963). There are two persons **A,B** with utility functions defined over commodities (x, y) as follows: $U_A(x, y) = \min(x, 2y)$ and $U_B(x, y) = \min(2x, y)$; their endowments are specified by $w_A = (1, 0), w_B = (0, 1)$; routine computations lead to the excess demand function of the first good (x), $Z(p)$, where p is the relative price of good x:

$$Z(p) = \frac{p - 1}{(p + 2)(2p + 1)}$$

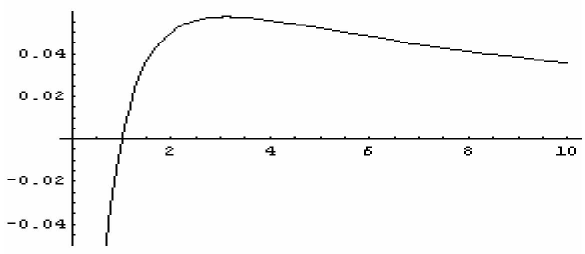
Thus the unique **interior** equilibrium is given by $p = 1$[10]; now notice that if the adjustment on prices is given by

$$\dot{p} = h(p) \tag{1}$$

where $h(p)$ has the same sign as $Z(p)$ and is continuously differentiable so that the solution to (1) say $p_t(p^o)$ is well defined for any initial point $p^o > 0$.

[10] There is an equilibrium at infinity; see for example, Figure 2.

Figure 2: Excess Demand - The Gale Example



As Gale (1963) says, " Arrow and Hurwicz have shown that for the case of two goods, one always has global stability...... Nevertheless, some queer things can happen even in this case." To see the queer things referred to, consider the function $V(p_t) = (p_t - 1)^2$ and notice that along the solution to the equation (1), we have $\dot{V}(t) > 0$ for all t, if $p^o \neq 1$: so that the price moves further away from equilibrium and there is no tendency to approach the unique equilibrium. Consequently, even for two goods or the single market case, we need to identify conditions which will help us rule out such cases.

Alternatively, a look at Figure 2 should convince us that the unique interior equilibrium is unstable under the usual adjustment of prices. How do we rule out such situations ?

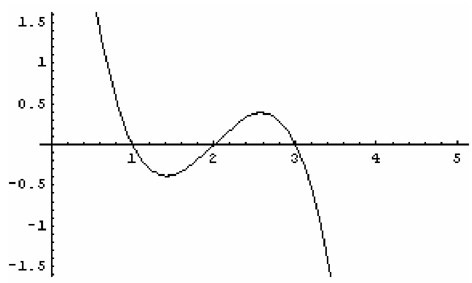
As an illustration of our analysis below, consider the following assumptions for the single market (two-good case)[11]:

α: $Z_x(p), Z_y(p)$ are continuous functions of p for all $p > 0$ and satisfies Walras Law viz., $pZ_x(p) + Z_y(p) = 0 \forall p > 0$; further $\lim_{p\to 0} Z_x(p) > 0$; similarly, $\lim_{p\to\infty} Z_y(p) > 0$.

It is easy to check that condition α implies that the set of equilibria $E = \{p >$

[11] We shall write $Z_x(p), Z_y(p)$ as the excess demand functions for the two goods x, y respectively with p being the relative price of good x.

Figure 3: Excess Demand under α and β



$0 : Z_x(p) = 0\}$ is non-empty[12]. Moreover, if $Z_x(p)$ is differentiable, then there must exist **at least one** $p^\star \in E$ such that $Z_x'(p^\star) < 0$: that is there must be **at least one** locally stable equilibrium. For global stability considerations however, we must proceed further.

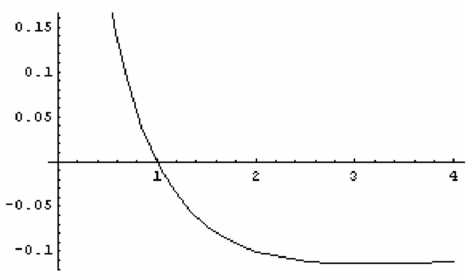
β: $p \in E \Rightarrow Z_x'(p) \neq 0$.

With conditions α, β however, we can show the following: first of all, there are only a finite number of points in E and secondly, for any $p^o > 0$, $p_t(p^o) \to p^\star \in E$ for some $p^\star \in E$: this may be considered to be the counter-part of global stability when there are multiple equilibria. Thus the situation will be as in Figure 3, then. **Thus one may say that overall the excess demand curve is downward sloping: since it begins from somewhere near the vertical axis and ends up below the horizontal axis.**

It should be noted that the boundary condition in α enforces the fact that excess demand functions have a negative slope most of the time; this is enough for global stability. We also know that the negative slope of excess demand function may be related to a Law of Demand; thus this condition, that excess demand functions be *mainly* downward sloping, may seem to be a generalization. We shall investigate below whether it will be possible to extend these conditions to cover

[12] See, for instance Mukherji (2002), p. 16.

Figure 4: Excess Demand for the Gale Example with a switch in endowments



the case when there are two or more markets.

There is one other aspect of the Gale example we should note; suppose for example, we interchange the endowments i.e., **A** has $(0, 1)$ while **B** has $(1, 0)$; recomputing excess demand functions, we note that the situation is as in Figure 4: the unique interior equilibrium is now globally stable. One may therefore say that we had instability of the interior equilibrium because the distribution of purchasing power had not been *right* previously. With the new pattern of endowments, both α, β are met and excess demand curve becomes downward sloping.

3 General Global Stability Conditions

Consider the following systems of equations:

$$\dot{x} = f(x, y) \text{ and } \dot{y} = g(x, y) \tag{2}$$

where the functions f, g are assumed to be of class $\mathcal{C}^1$ on the plane $\Re^2$. For any pair of functions $f(x, y), g(x, y)$ let $J(f, g)$ or simply J, if the context makes it clear, stand for the Jacobian[13]:

[13] f_x for any function f will refer to the partial derivative of f with respect to the variable x.

$$\begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}$$

We shall write[14] $\mathbf{div}(f, g)$ for the trace of $J(f, g) = f_x + g_y$. Consider, next, the following conditions:

- O1: There is an unique equilibrium $(\bar{x}, \bar{y})$ to (2).
- O2: $\mathbf{div}(f, g) = f_x + g_y < 0$ for all $(x, y) \in \Re^2$.
- O3: Determinant of $J(f, g) = \det J(f, g) = f_x.g_y - f_y.g_x > 0$ for all $(x, y) \in \Re^2$
- O4: Either $f_x.g_y \neq 0$ for all $(x, y) \in \Re^2$ or $f_y.g_x \neq 0$ for all $(x, y) \in \Re^2$.

Under O1 - O4, Olech's Theorem, Olech (1963), shows that the unique equilibrium $(\bar{x}, \bar{y})$ is *globally asymptotically stable.* The contribution in Ito (1978) provides conditions which, in addition, guarantee that the solution remains positive, a requirement which has great importance and relevance to economic theory[15].

We shall use the setting of the tatonnement to investigate motion on the plane and for this purpose we introduce the notion of the excess demand functions $Z_i(p_1, p_2, p_3) : \Re^3_{++} \to \Re, i = 1, 2, 3$ which are required to satisfy the following:
A. Each $Z_i(.)$ is continuously differentiable with continuous partial derivatives and is bounded from below on $\Re^3_{++}$; further for any $(p_1, p_2, p_3) \in \Re^3_{++}$, $p_1.Z_1(.) + p_2.Z_2(.) + p_3.Z_3(.) = 0$ (Walras Law); and further for any $(p_1, p_2, p_3) \in \Re^3_{++}$, $\forall i, Z_i(\lambda p_1, \lambda p_2, \lambda p_3) = Z_i(p_1, p_2, p_3)$ for any $\lambda > 0$ (Homogeneity of degree zero in the prices); finally, for any sequence, $P^s = (p_1^s, p_2^s, p_3^s) \in \Re^3_{++}$, $p_i^s = 1, \forall s$ for some index i, say $i = i_o$ and $||P^s|| \to +\infty$ as $s \to +\infty \Rightarrow Z_{i_o}(P^s) \to +\infty$[16](Boundary Condition).

[14] This is usually defined to be the **divergence** of (f, g).

[15] We must mention in this connection, the paper by Sasakura (1992), which considers the modifications required to ensure that that the solution to such a planar system remains within a bounded region.

[16] $||x||$ stands for $\sqrt{(x_1^2 + x_2^2 + x_3^2)}$, when $x = (x_1, x_2, x_3)$.

The conditions listed under **A** are all routine; however they do imply some consequences of interest. First of all under these conditions, the set of equilibria for the economy $E = \{p \in \Re^3_{++} : Z_i(p) = 0 \forall i\} \neq \emptyset$; an independent demonstration of this assertion would follow as a by product of the analysis of the dynamics.

To study the dynamics on the plane, we shall investigate the solutions to a system of equations of the following type:

$$\dot{p_i} = h_i(p), i = 1, 2 \text{ with } p_3 \equiv 1 \tag{3}$$

where the functions $h_i(p)$ are assumed to be continuously differentiable in the positive quadrant and satisfy the following: (we write $p = (p_1, p_2) \in \Re^2_{++}$)

B. $h_i(p)$ has the same sign $Z_i(p, 1)$ i.e., $h_i(p)$ is $+, -$ or 0 according as $Z_i(p, 1)$ is $+, -, 0$ $\forall p \in \Re^2_{++}, i = 1, 2$. The trajectory or solution to (3) from an initial $p^o \in \Re^2_{++}$, denoted by $\phi_t(p^o)$, remains within a bounded region of $\Re^2_{++}$.

Thus the equation (3) defines motion on the positive quadrant of the plane. The price configuration will be $(\phi_t(p^o), 1)$ for each instant t; this is just to signify that the numeraire (the third good) price is always kept fixed at unity. Also we note that any equilibrium for the dynamical system (3), say $\bar{p}$ where $h_i(\bar{p}) = 0, i = 1, 2$, implies that $(\bar{p}, 1)$ is an equilibrium for the economy, in the sense that $(\bar{p}, 1) \in E$ and conversely. We shall denote the set of equilibria for (3) by $\tilde{E}$.

We are interested in the structure of the ω-limit set $L_\omega(p^o)$ i.e., the limit points of the trajectory $\phi_t(p^o)$ as $t \to +\infty$. On the plane, the structure of ***non-empty and compact*** ω-limit sets is known to be one of the following[17]:

i. Consists of a single equilibrium or

ii. Consists of one closed orbit or

iii. an union of equilibria and paths tending to them.

It is because this classification offers some hope of obtaining general results that we shall investigate this situation more closely. We need to guarantee that

[17] See, for instance, Andronov et. al. (1966), p. 362.

the solution remains within the positive quadrant, which was the main item of concern in Ito (1978), as we mentioned above; then we need to guarantee that the ω-limit sets are non-empty; this will be accomplished by ensuring that the solution or trajectories are **bounded**; if a meaningful set of conditions allow us to rule out possibilities listed at (ii) and (iii), we have then a stability result. The conditions Olech (1963) mentioned above contain one such set of conditions; these need to be refined a bit if we want to ensure positivity as has been indicated in Ito (1978). As should be apparent, even for motion on the plane, the requirements are fairly stringent.

It may be noted that

1 *For each $i = 1, 2$, there exists $\varepsilon_i > 0$ such that $Z_i(p_i, p_j, 1) > 0$ if $p_i \leq \varepsilon_i$ for any p_j, $j \neq i$, $j = 1, 2$*[18].

Given the above claim, note that any trajectory of (3), $\phi_t(p^o) = (p_1(t), p_2(t))$, say, where $p_i^o > \varepsilon_i, i = 1, 2$ satisfies $p_i(t) > \varepsilon_i$ for all $t > 0$. Thus the trajectory remains within the positive orthant. This is important enough to be noted separately.

2 *Given $\boldsymbol{A}$ and $\boldsymbol{B}$, the solution $\phi_t(p^o)$ from any $p^o > (0, 0)$ remains within the positive orthant for all t and remains bounded away from the axes.*

Remark 1 *It should be pointed out that the above is valid even for motion in higher dimensions; there is nothing which depends on $n = 2$.*

What we shall consider next, however, will make crucial use of the fact that $n = 2$. Note then that on the basis of the above claims and assumptions, there is a rectangular region $R = \{(p_1, p_2) : \varepsilon_i \leq p_i \leq M_i\}$ in the positive quadrant within which the solution gets trapped provided the initial $p^o \in R$. To ensure convergence, we impose the following conditions:

[18]See Mukherji (2005).

C i. There is some function $\theta(p) : \Re^2_{++} \to \Re^2$ which is differentiable and such that $\mathbf{div}(\theta(p)h_1(p), \theta(p)h_2(p))$ is not identically zero on R nor does it change sign on R.
C ii. On the set $\tilde{E}$, $\mathbf{div}(\theta(p)h_1(p), \theta(p)h_2(p)) \neq 0, \det J(\theta(p)h_1(p), \theta(p)h_2(p)) \neq 0$.

Remark 2 *Keenan and Rader (1985) termed the assumption* $\mathbf{div}(Z_1, Z_2) < 0$ *as a Weak Law of Demand. Notice that this implies* **C i** *, when* $h_i = Z_i$ *and* $\theta(p) = 1$. *Our requirement is much weaker.*

Note that $\mathbf{div}(\theta(p)h_1, \theta(p)h_2) = \theta(p)\mathbf{div}(h_1, h_2) + \nabla\theta(p)(h_1, h_2)$[19]. Thus, **C i** allows for some change in the sign of $\mathbf{div}(h_1, h_2)$. In fact, we shall provide an interesting example where $h_i = Z_i$, $\mathbf{div}(h_1, h_2)$ is not of one sign and there exists a function $\theta(p)$ such that **C i** holds. It is of some importance to allow for such changes in the sign in $\mathbf{div}(Z_1, Z_2)$ since it is not expected that the Law of Demand (demand varies inversely with respect to own price) will hold for excess demand functions even at equilibria[20]. Notice also that **C i** does not impose any sign but restricts the extent to which sign changes can take place. It is for this reason that we may refer to **C i** as a Generalized Law of Demand.

C ii applies to equilibria. Once again no specific signs are imposed. We have:

3 *By virtue of* **C ii** *, the equilibria for the system (3) are either foci, nodes or saddle-points*[21]*; consequently, the number of equilibria are finite and odd*[22].

[19] $\nabla\theta(p)$ denotes the gradient vector of $\theta(p)$.

[20] Since net sellers' income effects may dominate.

[21] See, for instance Andronov et. al. (1966) p. 301 or Guckenheimer and Holmes (1983) p. 51. Consider the characteristic roots of the Jacobian evaluated at equilibrium. A focus is an equilibrium or fixed point with the characteristic roots are complex conjugates; the equilibrium is a stable focus when the real parts of these roots are negative; it is an unstable focus when the real parts are both positive; the equilibrium is called a node when these characteristic roots are both real and of the same sign; again it is a stable node if the real roots are both negative and an unstable node if the real roots are positive. Sometimes stable foci and nodes are called sinks; unstable nodes and foci are called sources. A saddle-point is an equilibrium when the characteristic roots are both real but of opposite sign.

[22] See Mukherji (2005)

Thus **C ii** ensures that equilibria are isolated and are not centers[23]. Thus these restrictions serve to ensure that small parameter changes do not affect either the static or the dynamic properties of equilibrium.

We have, further[24],

Proposition 1 *Under* $\mathbf{A}, \mathbf{B}$ *and* $\mathbf{C}$*, for any* $p^o \in R$*,* $L_\omega(p^o) = p^\star \in \tilde{E}$*. Thus all solutions converge to an equilibrium.*

We provide, next a set of remarks, which highlight and clarify the implications of the above result. First of all, it is easy to see that the conditions include meaningful cases, for instance:

Remark 3 *Consider the case when* $h_i(p) = Z_i(p, 1)$*: this is the standard case of tatonnement which is usually investigated. First of all, in this context, Keenan and Rader (1985) have investigated the implications of the Weak Law of Demand (viz.,* $\mathbf{div} Z(p, 1) = Z_{11}(p, 1) + Z_{22}(p, 1) < 0$*), when equilibria are isolated. Notice that these results are covered by the choice of* $\theta(p) = 1$*.*

The result provides a set of conditions under which an adjustment on prices on disequilibrium, in the direction of excess demand, will always lead to an equilibrium. It should also be pointed out that these conditions also imply the existence of a locally stable equilibrium or a sink.

Remark 4 *There will always be at least one sink i.e., an equilibrium at which the Jacobian has characteristic roots with real parts negative. To see this note that if no such equilibrium existed, then the only equilibria are saddle-points and sources. Also they are finite in number and moreover, as argued above, no trajectory can come close to sources; so the only possibility for a limit is a saddle-point; but each*

[23] If the characteristic roots of the Jacobian evaluated at equilibrium are pure complex roots, the equilibrium is a center.

[24] Mukherji (2005).

saddle-point has only one trajectory leading to it and there are an infinite number of possible trajectories, while there are a finite number of equilibria. Thus there must be a sink.

More importantly:

Proposition 2 *Under **A**, **B** and **C**, if there is a unique equilibrium, it must be globally asymptotically stable.*

Finally the implications of our proposition for a general system such as (2)may be noted. Assume that the set of equilibria for this system $E = \{(x, y) : f(x, y) = 0, g(x, y) = 0\}$ is non-empty. The following may be noted:

Proposition 3 *If*
i. There is a rectangular region $R = \{(x, y) : 0 \leq x \leq M, 0 \leq y \leq N\}$ *such that any trajectory of (2) on the boundary of* R *is either inward pointing or coincides with the boundary;*
ii. There is a function $\theta(x, y)$ *such that* $\mathbf{div}(\theta(.)f(.), \theta(.)g(.))$ *is not identically zero on* R *nor does it change sign on* R.
iii. On the set E, $\mathbf{div}(\theta(.)f(.), \theta(.)g(.)) \neq 0, \det J(\theta(.)f(.), \theta(.)g(.)) \neq 0$;
then any trajectory $\phi_t(x^o, y^o)$ *where* $(x^o, y^o) > (0, 0)$ *converges to a point of* E.

4 An Application: The Scarf Example

Consider[25] an exchange model where there are three individuals $h = 1, 2, 3$ and three goods $j = 1, 2, 3$. The utility functions and endowments are as under:

$$U^1(q_1, q_2, q_3) = min(q_1, q_2);\ w^1 = (1, 0, 0)$$

$$U^2(q_1, q_2, q_3) = min(q_2, q_3);\ w^2 = (0, b, 0)$$

[25] We provide an analysis of the Scarf example which is somewhat different from the one in Scarf (1960). In particular, it should be pointed out that Scarf did not use a numeraire.

$$U^3(q_1, q_2, q_3) = min(q_1, q_3);\ \ w^3 = (0, 0, 1)$$

Routine calculations lead to the following excess demand functions, where good 3 is treated as numeraire (i.e., $p_3 = 1$): Note that $b = 1$ would take us to the example considered by Scarf (1960).

We treat good 3 as the numeraire and then compute excess demand functions for the non-numeraire commodities for the case at hand; it turns out that these are given, using the same notation as above, by the following expressions:

$$Z_1(p_1, p_2) = \frac{p_1(1 - p_2)}{(1 + p_1)(p_1 + p_2)}$$

$$Z_2(p_1, p_2) = \frac{p_2(p_1 - b) + (1 - b)p_1}{(1 + p_2)(p_1 + p_2)}$$

Consequently the system (**??**) now takes the form:

$$\dot{p_1} = \frac{p_1(1 - p_2)}{(1 + p_1)(p_1 + p_2)} \text{ and } \dot{p_2} = \frac{p_2(p_1 - b) + (1 - b)p_1}{(1 + p_2)(p_1 + p_2)} \tag{4}$$

Once more standard computations ensure that the **unique equilibrium** is given by

$$p_1^* = \frac{b}{2 - b} = \theta \text{ say}, p_2^* = 1$$

Thus it may be noted that our choice of the parameter places a restriction on its magnitude

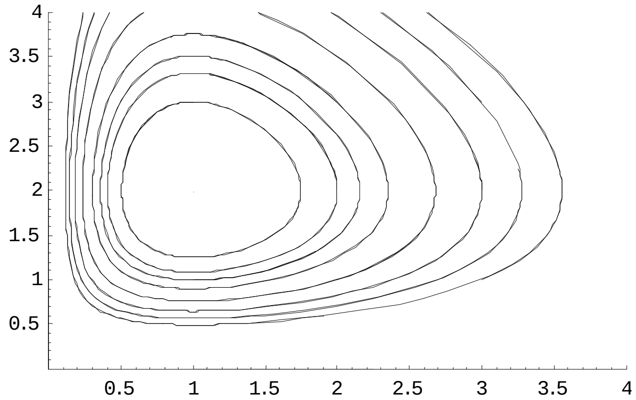
$$0 < b < 2;$$

and we shall take it that this is met. Notice also that when $b = 1$, $\theta = 1$ too, and we have the equilibrium depicted in Scarf (1960). Notice first of all from Figure 5, that for $b = 1$, the solution follows closed orbits around equilibrium and there is no convergence.

If $b \neq 1$, there are some changes to the stability property of equilibrium is evident from computing characteristic roots :[26] Some tedious calculations reveal

[26]In fact it was shown in Mukherji (2000) that $b = 1$ provides a point of Hopf Bifurcation for the process (4).

Figure 5: The Scarf Example



that the characteristic roots of the relevant matrix at equilibrium are given by:

$$\frac{1}{8}(-b+b^2 \pm \sqrt{b}\sqrt{\{-32+49b-26b^2+5b^3\}}). \tag{5}$$

Consequently, one may claim:

4 *For the process (4), $(\theta, 1)$ is a locally asymptotically stable equilibrium if and only if $b < 1$; for $b > 1$, the equilibrium is locally unstable.*

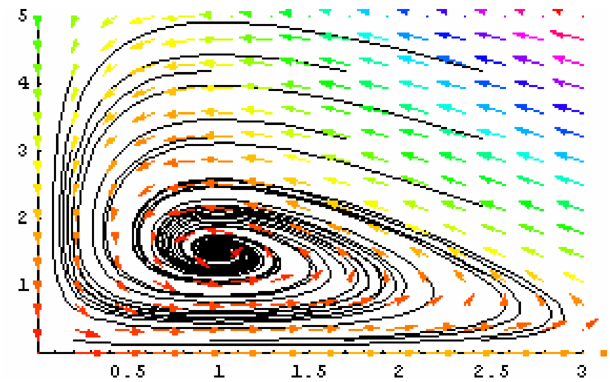
A much stronger assertion is possible[27]:

5 *For the system (4), the unique equilibrium $(\theta, 1)$ is globally asymptotically stable whenever $b < 1$; and any trajectory with $(p_1^o, p_2^o) > (0,0)$ as initial point remains within the positive orthant. When $b > 1$, any solution with an arbitrary non-equilibrium initial point is unbounded.*

For $b < 1$ the solutions are as in Figure 6; proofs of this and the other assertions may be found in Mukherji (2005).

[27]See Mukherji (2002), p. 89-90. We provide an alternative approach as an application of our result obtained in this paper.

Figure 6: A Perturbation of the Scarf Example



There are thus two things to be noted from the above: first, choosing a value of b **different** from unity negates the existence of a closed orbit; and a value of b **less than** unity is required to ensure that trajectories remain bounded. As we have shown, these are the only two aspects we need to account for if we are interested in identifying global stability conditions for motion on the plane.

5 Conclusion

The above analysis shows, first of all, that cyclical behavior around equilibrium, noted by Scarf, is not robust particularly with reference to perturbation of the endowments. In an identical set-up, results exist which show that a redistribution of the goods among individuals may also help to restore stability to the Scarf example: Hirota (1981) and Hirota (1985) both contain illuminating results in this connection[28].

[28]The interesting contribution in Fujimoto (1990) looks at a slightly different question. It is shown that a different price mechanism is able to attain equilibrium for the Scarf example; the price mechanism is discrete and considers a weighted average of past prices, together with the current level of excess demand in determining revised prices.

More importantly, in the realm of micro-economic theory, it is well known that the substitution effects are all in the proper direction and only income effects may ruin stability. This intuition while being relevant for 'local' stability analysis is not of much help when we try to understand global stability of equilibrium. The Scarf (as well as the Gale) example is an effort at ruling out all substitution effects; the instability noted by Scarf might then have been assumed to be due to this fact; that this is not the case, may be seen by our result, since without introducing any substitution effects, the economy has a globally stable equilibrium when $b < 1$.

A more recent paper Anderson et. al. (2004), points out the existence of an endowment distribution which leads to global stability. The identified endowment distribution is the one where each individual has an unit of the good that he is not interested in[29]: that is individual one has a unit of good 3; individual 2 has an unit of good 1 and individual 3 has an unit of good 2. The important and significant part of the contribution made in Anderson et. al. (2004) lies in their discovery that experiments conducted with agents with similar preferences and endowments, but engaging in double auctions would lead to price movements which are predicted by the tatonnement model. Thus the results provided by the tatonnement process, they argue, should be looked at with greater care because they seem to predict what price adjustments might actually occur.

As we showed in our analysis of the Scarf Example, the perturbation allowed us to get rid of closed orbits; for convergence, we needed to show that the solution was bounded. One of the reasons for our being able to obtain such a different result was due to the fact that at the original equilibrium, the relevant matrix had purely complex characteristic roots, with zero real parts. It is not surprising that in such a situation, a perturbation changed the real parts of the characteristic root from zero to positive or negative.

[29]See in this connection, the example in Gale (1963) with two goods and two individuals with similar tastes discussed earlier.

In Hicks (1946), there is an enquiry relating to the following questions: if a market is stable by itself, can it be rendered unstable from the price adjustment in other markets ? Alternatively, if a market is unstable when taken by itself, can it be rendered stable by the price adjustment in the other markets ? To both an answer was provided in the negative. The conditions in **C** may be related to this query.

Consider the case when $h_i(p) = Z_i(p, 1)$; then the restrictions are in terms of $\mathbf{div}(\theta(p)Z_1(p, 1), \theta(p)Z_2(p, 1)) =$
$\theta(p)\mathbf{div}(Z_1(p, 1), Z_2(p, 1)) + \nabla\theta(p)(Z_1(p, 1), Z_2(p, 1))$; (as before, $\nabla\theta(p)$ refers to the gradient vector of $\theta(p)$); notice that the phrase "stable by itself" refers to the terms $Z_{jj}(p, 1) < 0$ which in turn would make $\mathbf{div}(Z_1(p, 1), Z_2(p, 1)) < 0$ so that a choice of $\theta(p) = 1$ would satisfy the requirements in **C**. This however is a rather strong requirement; a weaker form would be to require that $\mathbf{div}(Z_1, Z_2) < 0$: this is what Kennan and Rader (1985) have called the Weak Law of Demand; our investigation shows that even this is not required; indeed , for the Scarf Example, this is violated; what happens when $b < 1$ is that there is function $\theta(p)$ such that $\mathbf{div}(\theta(p)Z_1(p, 1), \theta(p)Z_2(p, 1))$ is one-signed and that rules out cycles.

If we are to insist that

$$\frac{\partial Z_i}{\partial p_i} < 0 \ \forall \ i$$

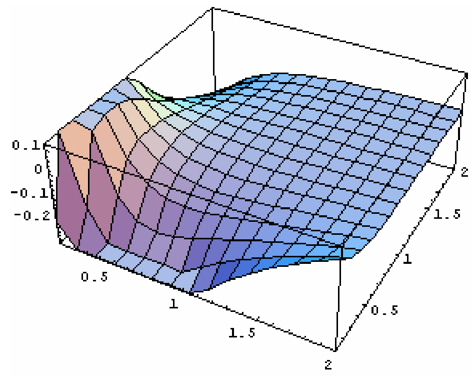
then that would be like insisting on the excess demand curves are downward sloping everywhere; a weaker requirement would be to insist on

$$\mathbf{div}(Z_1, Z_2) = \sum_{i=1}^{2} \frac{\partial Z_i}{\partial p_i} < 0$$

which is what Keenan and Rader (1985) called the Weak Law of Demand. Our condition **C** is weaker in that it requires the existence of functions $\theta(.), h_i(.)$ such that $h_i(.)$ has the same sign as excess demand Z_i for all i, and that

$$\mathbf{div}(\theta h_1, \theta h_2) \text{ has the same sign ;}$$

Figure 7: Violation of the Weak Law of Demand by the Scarf Example



it may be noted that for the Scarf example, even for $b < 1$, where $h_i = Z_i$, the Keenan and Rader (1985) restriction is not met (see, for instance, Figure 7; it will be noticed that the sign along the vertical axis changes sign):

while for the choice

$$\theta(p_1, p_2) = \frac{(1+p_1)(1+p_2)(p_1+p_2)}{p_1.p_2}$$

it follows that

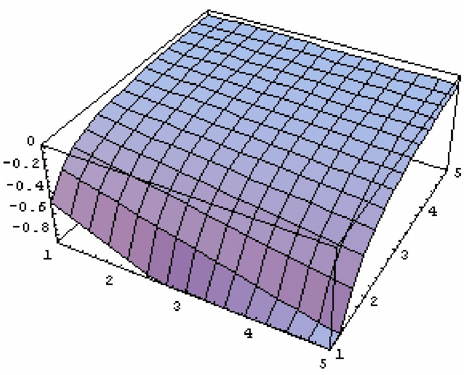
$$\mathbf{div}(\theta Z_1, \theta Z_2) = -\frac{(1-b)}{p_2^2}$$

which has the same sign for any value of $b \neq 1$; and thus closed orbits are ruled out whenever $b \neq 1$. In particular one may show that $b < 1$ the orbits are bounded and our convergence result follows. Thus notice that our definition allows for some variation in the sign of the slopes of excess demand curves; our condition, involving the choice of θ as above, is satisfied and the $\mathbf{div}(\theta h_1, \theta h_2)$ remains single signed, as depicted in Figure 8.

It is in this sense, that we would like to characterise the condition as one requiring that overall the excess demand curves are downward sloping.

Finally, there is another point that needs to be noted. During the course of

Figure 8: The General Law of Demand Satisfied by the Scarf Example



our analysis of the examples due to Gale and Scarf, it became evident that the distribution of endowments were not 'proper' for stability of equilibrium; in each case, altering the endowment distribution led to equilibrium being globally stable. Thus for the effectiveness of the Invisible Hand in Motion and more so for the effectiveness of policies depending on such matters, a proper income distribution (or more correctly, a proper distribution of purchasing power) seems to be required.

References

C. M. Anderson, S. Granat, C. R. Plott, K-I. Shimomura, (2004), Global Instability in Experimental General Equilibrium: The Scarf Example, J. Econ. Theory, 115, 209-249.

A. A. Andronov, A. A. Vitt, S. E. Khaikin (1966), Theory of Oscillators, (translated from Russian by F. Immirzi; translation edited and abridged by W. Fishwick), Pergamon Press, Oxford.

K. J. Arrow and L. Hurwicz, (1958), On the Stability of Competitive Equilibrium I, Econometrica, 26, 522-552.

P. Champsaur, J. H. Dreze and C. Henry, (1977), Stability Theorems with Economic Applications Econometrica, Vol. 45, No. 2. p. 273-294.

C. d'Aspremont and J. H. Dreze (1979), On the Stability of Dynamic Processes in Economic Theory Econometrica, Vol. 47, No. 3, p. 733-738.

G. Debreu (1974), Excess Demand Functions, J. Math. Econ. 1 (1974), 15-21.

T. Fujimoto (1990), Stabilizing Scarf's Unstable Examples, Okayama Economic Review, 22 67-71.

D. Gale (1963), A Note on the Global Instability of Competitive Equilibrium, Naval Research Logistics Quarterly, 10 81-87.

J. Guckenheimer, P. Holmes (1983), Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields, Springer-Verlag, Berlin.

F. H. Hahn (1984), Equilibrium and Macroeconomics, Basil Blackwell, Oxford.

J. R. Hicks (1946), Value and Capital (2nd Edition), Clarendon Press, Oxford.

M. Hirota (1981), On the Stability of Competitive Equilibrium and the Patterns of Initial Holdings: An Example, Int. Econ. Rev. 22, 461-470.

M. Hirota (1985), Global Stability in a Class of Markets with Three Commodities and Three Consumers, J. Econ. Theory 36 , 186-192.

M. Hirota, M. Hsu, C. Plott and B. Rogers (2005), Divergence, Closed Cycles and Convergence in Scarf Environments: Experiments in the Dynamics of General Equilibrium Systems, Caltech Social Science Working Paper, 1239, October 2005.

M. W. Hirsch, S. Smale (1974), Differential Equations, Dynamical Systems and Linear Algebra, Academic Press, New York.

T. Ito (1978), A Note on the Positivity Constraint in Olech's theorem, Journal of Economic Theory, 17, 312-318.

D. Keenan and T. Rader (1985), Market Dynamics and the Law of Demand, Econometrica, 53 , 465-472.

L. W. McKenzie, (1960), Stability of Equilibrium and the Value of Positive Excess Demand, Econometrica, 28, 606-617.

L. W. McKenzie, (2002), Classical General Equilibrium Analysis, Princeton University Press, New Jersey.

A. Mukherji, (1974), Stability in an Economy with Production, in Trade, Stability and Macroeconomics, Essays in honor of Lloyd Metzler eds., G. Horwich and P. A. Samuelson, Academic Press, New York, 243-258.

A. Mukherji, (1989), On Tatonnement Processes, Centre for Economic Studies and Planning Working Paper, Jawaharlal Nehru University, New Delhi.

A. Mukherji, (1990), A Non-Tatonnement Process, in Economic Theory and Policy: Essays in honor of Professor D. Banerjee, (eds.), Debraj Ray et. al., Oxford University Press: Delhi.

A. Mukherji, (1995), A Locally Stable Adjustment Process, Econometrica, 63, 441-448.

A. Mukherji (2000), Non-linear Dynamics with applications to economics: Stability of Competitive equilibrium reconsidered, Journal of Quantitative Economics 16 , 93-144.

A. Mukherji (2002), An Introduction to General Equilibrium Analysis, Walrasian and Non-Walrasian Equilibria, Oxford University Press, Delhi.

A. Mukherji (2005), Global Stability Conditions on the Plane: A General Law of

Demand, Centre for Economic Studies and Planning Working Paper, Jawaharlal Nehru University, New Delhi.

T. Negishi (1962), Stability of a Competitive Economy, Econometrica, 30, 635-669.

C. Olech (1963), On the Global Stability of an Autonomous System on the Plane, Contributions to Differential Equations 1 , 389-400.

P. A. Samuelson (1947), Foundations of Economic Analysis, Harvard University Press: Cambridge, Mass.

K. Sasakura, (1992), Boundedness of Economic Variables and Olech's Theorem, Journal of Economics, Zeitschrift für Nationalökonomie, 56, 209-217.

H. Scarf (1960), Some Examples of Global Instability of the Competitive Equilibrium, Int. Econ. Rev. 1 , 157-172.

A. K. Sen (1999), Development as Freedom, Oxford University Press, Oxford.

S. Smale (1976), On the Differential Equations of Species in Competition, Journal of Mathematical Biology, 3, 5-7.

H. Sonnenschein (1972), Market Excess Demand Functions, Econometrica, 40 , 549-563.